Session 4

The proportional odds model and its application

4.1 Proportional odds model (POM) overview

4.2 Proportional odds model theory

4.3 Application of the POM using SAS

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4.1 Proportional odds model (POM) overview

Proportional Odds Modelling

Is a way of modelling outcome when the outcome has more than two ordered categories and is not continuous.

tSAH – Study A

	Good	Moderate	Severe	Dead/Veg
No	247	100	73	88
Yes	113	73	65	147

Dichotomies

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Odds Ratios

	tSAH – Study A
Not Good	2.39
Unfavourable	2.46
Dead/Veg	2.80
Proportional odds model	2.51

4.2 Proportional odds model theory

Notation

Category	Control Group	Treated Group	Total
C_1	n_{1C}	n_{1T}	n_1
C_2	n_{2C}	n_{2T}	n_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
C_m	n_{mC}	n_{mT}	n_m
Total	n_C	n_T	n

(i) POM for two groups

Assume that

$\theta_1 = \theta_2 = \ldots = \theta_{m-1}$

and denote the common value by θ

Thus $$\theta = \log\left[\frac{Q_{kT}\left(1-Q_{kC}\right)}{Q_{kC}\left(1-Q_{kT}\right)}\right] \qquad k=1,\ldots,m-1$$

(ii) General POM

Subjects in the study are

$S_1, S_2, \ldots, S_n$

Subject S_i has covariate

$\underline{z}_i' \equiv (z_{i1}, z_{i2}, \ldots, z_{iq})$

and probability distribution

C_1	C_2	…………	C_m
$p_1(\underline{z}_i)$	$p_2(\underline{z}_i)$	…………	$p_m(\underline{z}_i)$

$$Q_k(\underline{z}_i) = P(C_k \text{ or better}; \underline{z}_i)$$

$$= p_1(\underline{z}_i) + \ldots + p_k(\underline{z}_i)$$

The linear model is defined by

$$\log\left[\frac{Q_k(\underline{z}_i)}{1-Q_k(\underline{z}_i)}\right] = \alpha_k + \eta_k(\underline{z}_i) \qquad k=1,\ldots,m-1$$

where

α_k = k^{th} intercept

$\eta_k(\underline{z}_i) = \underline{z}_i'\underline{\beta}_k = z_{i1}\beta_{k1} + \ldots + z_{iq}\beta_{kq}$

= Linear predictor

Assume that

$\underline{\beta}_1 = \underline{\beta}_2 = \ldots = \underline{\beta}_{m-1} = \underline{\beta}$

so that: $$\eta_1(\underline{z}_i) = \eta_2(\underline{z}_i) = \ldots = \eta_{m-1}(\underline{z}_i) = \eta(\underline{z}_i)$$

i.e. $$\eta(\underline{z}_i) = \underline{z}_i'\,\underline{\beta} = z_{i1}\beta_1 + \ldots\ldots + z_{iq}\beta_q$$

Hence: $$\log\left[\frac{Q_k(\underline{z}_i)}{1 - Q_k(\underline{z}_i)}\right] = \alpha_k + \eta(\underline{z}_i)$$

Proportional odds model
McCullagh (1980)

To compare subject S_i with baseline subject:

S_0: covariates $\underline{0}$

S_i: covariates $\underline{z}_i$

$$\log\left[\frac{Q_k(\underline{z}_i)\{1-Q_k(\underline{0})\}}{Q_k(\underline{0})\{1-Q_k(\underline{z}_i)\}}\right]=(\alpha_k+\eta(\underline{z}_i))-(\alpha_k+\eta(\underline{0}))=\eta(\underline{z}_i)$$

log odds ratio relative to baseline subject

If $\eta(\underline{z}_i) > 0$, S_i 'better' than baseline subject

$\eta(\underline{z}_i) < 0$, S_i 'worse' than baseline subject

To compare subject S_i relative to subject S_h

$$\log\left[\frac{Q_k(\underline{z}_i)\{1-Q_k(\underline{z}_h)\}}{Q_k(\underline{z}_h)\{1-Q_k(\underline{z}_i)\}}\right] = (\alpha_k + \eta(\underline{z}_i)) - (\alpha_k + \eta(\underline{z}_h)) = \eta(\underline{z}_i) - \eta(\underline{z}_h)$$

log odds ratio of S_i relative to S_h is

$$\eta(\underline{z}_i) - \eta(\underline{z}_h)$$

4.3 Application of the POM using SAS

Example 1: Head injury data – head1 (grouped data)

Glasgow Outcome Scale (GOS) at 3 months	Glasgow Coma Scale (GCS) on entry 3-5	6-8	Total
1. Good Recovery	73	219	292
2. Moderate Disability	55	118	173
3. Severe Disability	79	66	145
4. Vegetative State	37	10	47
5. Dead	358	92	450
Total	602	505	1107

$$\text{gos} = \begin{cases} 1: & \text{Good Recovery} \\ 2: & \text{Moderate Disability} \\ 3: & \text{Severe Disability} \\ 4: & \text{Vegetative State} \\ 5: & \text{Dead} \end{cases}$$

$$\text{gcsentry} = \begin{cases} 1: & \text{GCS on entry } 3\text{-}5 \\ 2: & \text{GCS on entry } 6\text{-}8 \end{cases}$$

Use SAS Proc Logistic to fit:

$$\log\left[\frac{Q_k(\underline{z}_i)}{1-Q_k(\underline{z}_i)}\right]=\alpha_k+\eta(\underline{z}_i) \qquad k=1,\ldots,4$$

where $Q_k(\underline{z}_i)$ = P(GOS score k or better; $\underline{z}_i$)

$\eta(\underline{z}_i) = \beta z_i$

$$z_i = \begin{cases} 0: \text{ if gcsentry} = 1 \text{ (GCS 3-5)} \\ 1: \text{ if gcsentry} = 2 \text{ (GCS 6-8)} \end{cases}$$

For the Proportional odds model a SAS Proc Logistic program and output are shown in Supplement 4.1

Example 2: Head injury data – head2 (ungrouped data)

Glasgow Outcome Scale Count (%)	Treatment		Total
	Control	Treated	
1: Good recovery	42 (25)	71 (40)	113 (33)
2: Moderate disability	27 (16)	30 (17)	57 (17)
3: Severe disability	33 (20)	27 (15)	60 (18)
4: Vegetative state/Dead	63 (38)	48 (27)	111 (33)
Total	165 (100)	176 (100)	341 (100)

Objective: to relate

Outcome	GOS (4 categories)

to

Baseline age

GCS motor	1 = None or Extension
	2 = Abnormal or Normal Flexion
	3 = Localises
Treatment	0 = Control
	1 = Treated

For the Proportional odds model a SAS Proc Logistic program and output are shown in Supplement 4.2